Name: Final 2013 SOLUTION

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Multiple Choice

1.(6 pts.) Let $f(x) = \dfrac{e^x + 1}{1 - 2e^x}$. The inverse function $f^{-1}(x)$ is

(a) $\ln(x-1) - \ln(1+2x)$ (b) $\ln(1-x) - \ln 1 - 2x$

(c) $\ln(1-2x) - \ln(1+2x)$ (d) $\ln(1-2x) - \ln(1-x)$

(e) $\ln(x-1) - \ln(1-2x)$

To solve for the inverse, let's put $f(x) = x$ and $x = y$, we have then

$$x = \frac{e^y + 1}{1 - 2e^y} \Leftrightarrow x - 2xe^y = e^y + 1 \Leftrightarrow x - 1 = e^y + 2xe^y \Leftrightarrow x - 1 = e^y(1+2x)$$

$$\Leftrightarrow e^y = \frac{x-1}{1+2x} \Leftrightarrow y = \ln\left(\frac{x-1}{1+2x}\right) \Leftrightarrow y = \ln(x-1) - \ln(1+2x)$$

So, $\boxed{f^{-1}(x) = \ln(x-1) - \ln(1+2x)}$

2.(6 pts.) Let $f(x) = (3+x)e^{-x}$. Find $(f^{-1})'(3)$.

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) -1 (d) $-\frac{1}{2}$ (e) $\frac{1}{2}$

We have $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$. To find $f^{-1}(3)$, we let $f^{-1}(3) = b$

$\Rightarrow f(b) = 3$

$\Rightarrow (3+b)e^{-b} = 3$

$\Rightarrow b = 0$

So, $f^{-1}(3) = 0$ and $f'(x) = 1 \cdot e^{-x} + (3+x)e^{-x}(-1) = e^{-x} - (3+x)e^{-x}$

Then $f'(f^{-1}(3)) = f'(0) = e^{-0} - (3+0)e^{-0} = 1 - 3 = -2$

Hence, $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))}$

$= \boxed{-\frac{1}{2}}$

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3.(6 pts.) Evaluate the derivative of

$$f(x) = 2\arctan(\arcsin(\sqrt{x})).$$

(Recall: $\arctan y = \tan^{-1} y$ and $\arcsin y = \sin^{-1} y$.)

(a) $\dfrac{-1}{\sqrt{x-x^2}(1+\arcsin(x))}$

(b) $\dfrac{-1}{2\sqrt{x-x^2}(1+[\arcsin(\sqrt{x})]^2)}$

~~(c)~~ $\dfrac{1}{\sqrt{x-x^2}(1+[\arcsin(\sqrt{x})]^2)}$

(d) $\dfrac{-1}{\sqrt{1-x}(1+\arcsin(\sqrt{x}))}$

(e) $\dfrac{1}{2\sqrt{x-x^2}(1+\arcsin(\sqrt{x}))}$

$$f'(x) = 2\frac{1}{1+(\arcsin(\sqrt{x}))^2} \cdot \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}\frac{1}{\sqrt{x}}$$

$$= \frac{1}{\sqrt{1-x}\sqrt{x}\,(1+[\arcsin(\sqrt{x})]^2)} = \frac{1}{\sqrt{x-x^2}\,(1+[\arcsin(\sqrt{x})]^2)}$$

4.(6 pts.) Evaluate the limit

$$\lim_{x\to 0}(1-\sin(x))^{\frac{1}{2x}}. = L$$

(a) e^2 (b) $\frac{1}{e}$ (c) 1 ~~(d)~~ $e^{-\frac{1}{2}}$ (e) e

$$\lim_{x\to 0}(1-\sin(x))^{\frac{1}{2x}} = L \Rightarrow \lim_{x\to 0}\left[\ln(1-\sin(x))^{1/2x}\right] = \ln L$$

$$\Rightarrow \lim_{x\to 0} \underbrace{\frac{1}{2x}\left(\ln(1-\sin x)\right)}_{\left(\frac{0}{0} \to \text{use L'H rule}\right)} = \ln L \Rightarrow \lim_{x\to 0}\frac{\frac{-\cos x}{1-\sin x}}{2} = \ln L$$

$$\Rightarrow \frac{-\cos 0}{2(1-\sin 0)} = \ln L \Rightarrow \frac{-1}{2} = \ln L \Rightarrow \boxed{L = e^{-1/2}}$$

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5.(6 pts.) Evaluate $\int_1^2 x \ln x dx$. Integration by parts

(a) $\ln(2) - 1$ (b) 4 (c) $\ln(2)$

~~(d)~~ $2\ln(2) - \frac{3}{4}$ (e) $4\ln(2) - 4$

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\int_1^2 x \ln x\,dx = \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x\,dx$$

$$= \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^2$$

$$= \left[\frac{1}{2}(2^2)\ln(2) - \frac{1}{4}(2^2)\right] - \left[0 - \frac{1}{4}(1)^2\right]$$

$$= 2\ln(2) - 1 + \frac{1}{4} = \boxed{2\ln(2) - \frac{3}{4}}$$

6.(6 pts.) Evaluate the integral

$$\int \tan^2\theta \sec^4\theta\, d\theta.$$

(Note: The formula sheet may help.)

~~(a)~~ $\dfrac{\tan^5\theta}{5} + \dfrac{\tan^3\theta}{3} + C$ (b) $\dfrac{\tan^4\theta}{4} + \dfrac{\tan^2\theta}{2} + C$

(c) $\dfrac{\tan^3\theta}{3} + C$ (d) $\dfrac{\sec^5\theta}{5} + C$

(e) $\dfrac{\sec^5\theta}{5} + \dfrac{\sec^3\theta}{3} + C$

Let $u = \tan\theta \Rightarrow du = \sec^2\theta\, d\theta$. So, $\int \tan^2\theta \sec^4\theta\, d\theta = \int \tan^2\theta \sec^2\theta(\sec^2\theta\, d\theta)$

$$= \int \tan^2\theta(1+\tan^2\theta)(\sec^2\theta\, d\theta) = \int u^2(1+u^2)\,du = \int u^2 + u^4\, du$$

$$= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \boxed{\frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C}$$

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7.(6 pts.) Evaluate the integral

$$\int \frac{1}{x^2(x^2+1)}\,dx.$$

(Recall: $\arctan x = \tan^{-1} x$ and $\arcsin x = \sin^{-1} x$.)

Note that $\frac{1}{x^2(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$

(or do partial fraction : $\frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$ then $A=0$, $B=1$, $C=0$ $D=-1$)

(a) $\ln x + \frac{1}{x} + \arctan x + C$

(b) $\frac{1}{x} - \arctan x + C$

~~(c)~~ $\frac{-1}{x} - \arctan x + C$

(d) $\ln(x^2) + \arcsin x + C$

(e) $\frac{1}{x} + \ln(x^2+1) + C$

So, $\int \frac{1}{x^2(x^2+1)}\,dx = \int \frac{1}{x^2}\,dx - \int \frac{1}{x^2+1}\,dx = \boxed{\frac{-1}{x} - \arctan(x) + C}$

8.(6 pts.) Which of the following gives the trapezoidal approximation with $n = 6$ to the integral

$$\int_0^3 \underbrace{e^{(x^2)}}_{= f(x)}\,dx?$$

$a=3$, $b=0$

$\Delta x = \frac{b-a}{n} = \frac{3}{6} = \frac{1}{2}$

~~(a)~~ $\frac{1}{4}\left[1 + 2e^{1/4} + 2e + 2e^{9/4} + 2e^4 + 2e^{25/4} + e^9\right]$

(b) $\frac{1}{4}\left[1 + 4e^{1/4} + 2e + 4e^{9/4} + 2e^4 + 4e^{25/4} + e^9\right]$

(c) $\frac{1}{6}\left[1 + 4e^{1/4} + 2e + 4e^{9/4} + 2e^4 + 4e^{25/4} + e^9\right]$

(d) $\frac{1}{2}\left[1 + e^{1/4} + e + e^{9/4} + e^4 + e^{25/4} + e^9\right]$

(e) $\frac{1}{2}\left[1 + 2e^{1/4} + 2e + 2e^{9/4} + 2e^4 + 2e^{25/4} + e^9\right]$

$$\int_0^3 e^{x^2}\,dx = \frac{\Delta x}{2}\left[f(0) + 2f(\tfrac{1}{2}) + 2f(1) + 2f(\tfrac{3}{2}) + 2f(2) + 2f(\tfrac{5}{2}) + f(3)\right]$$

$$= \frac{1}{4}\left[1 + 2e^{1/4} + 2e + 2e^{9/4} + 2e^4 + 2e^{25/4} + e^9\right]$$

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9.(6 pts.)Determine whether the following integral converges or diverges. If it converges, evaluate it.

$$\int_{-2}^{3} \frac{2}{x^3} dx. \quad = \int_{-2}^{0} \frac{2}{x^3} dx + \int_{0}^{3} \frac{2}{x^3} dx$$

(a) 1

(b) $\frac{13}{36}$

~~(c)~~ The integral diverges.

(d) $2 \ln\left(\frac{27}{8}\right)$

$$\int \frac{2}{x^3} dx = -\frac{1}{x^2} + C$$

(e) $\frac{5}{36}$

$$\int_{-2}^{0} \frac{2}{x^3} dx = \lim_{t \to 0^-} \int_{-2}^{t} \frac{2}{x^3} dx = \lim_{t \to 0^-} \left[-\frac{1}{x^2} \right]_{-2}^{t} = \lim_{t \to 0^-} \left[-\frac{1}{t^2} - \frac{1}{(-2)^2} \right] = -\infty$$

$-\infty$

So, $\int_{-2}^{3} \frac{2}{x^3} dx$ diverges.

10.(6 pts.)Evaluate the integral

$$\int_{2}^{\infty} \frac{1}{x(\ln x)^2} dx.$$

(a) 0

(b) The integral diverges.

(c) $-\ln 2$

~~(d)~~ $\frac{1}{\ln 2}$

(e) $\frac{1}{2}$

$u = \ln x$

$du = \frac{1}{x} dx$

$$\int_{u=\ln 2}^{\infty} \frac{1}{u^2} du = \lim_{t \to \infty} \left[\frac{-1}{u} \right]_{\ln 2}^{t} = \lim_{t \to \infty} \left[-\frac{1}{t} + \frac{1}{\ln 2} \right]$$

0

$$= 0 + \frac{1}{\ln(2)} = \boxed{\frac{1}{\ln(2)}}$$

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11.(6 pts.)Compute the distance covered by a particle moving along the curve $y = \frac{2}{3}x^{3/2}$ from the point $(0, 0)$ to the point $(4, \frac{16}{3})$.

(a) $\frac{14}{3}$ (b) $\frac{1}{5\sqrt{5}}$ (c) $\ln 4$

~~(d)~~ $\frac{2}{3}(5\sqrt{5} - 1)$ (e) $(5\sqrt{5} - 1)$

$$y = \frac{2}{3}x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{2}{3}\left(\frac{3}{2}\right)x^{1/2} = \sqrt{x}$$

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1 + x} \underset{du = dx}{\overset{u = 1+x}{=}} \int_1^5 u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_1^5$$

$$= \frac{2}{3}\left[5^{3/2} - 1\right] = \frac{2}{3}\left[5 \cdot 5^{1/2} - 1\right] = \boxed{\frac{2}{3}\left(5\sqrt{5} - 1\right)}$$

12.(6 pts.)Which of the following are the orthogonal trajectories to the family of curves $x^2 + 2y^2 = k$?

~~(a)~~ $y = cx^2$ (b) $y^2 - x^2 = c$

(c) $y^2 + x^2 = c$ (d) $x = \sqrt{2}\,y$

(e) $x = y + c$

$$\frac{d}{dx}\left(x^2 + 2y^2\right) = \frac{d}{dx}(k) \Leftrightarrow 2x + 4y\frac{dy}{dx} = 0 \Leftrightarrow \frac{dy}{dx} = \frac{-2x}{4y} \Leftrightarrow \frac{dy}{dx} = \frac{-x}{2y}$$

The dif. eq we need to solve is $\frac{dy}{dx} = \frac{-1}{\frac{-x}{2y}} = \frac{2y}{x}$

$$\Leftrightarrow \int \frac{dy}{2y} = \int \frac{dx}{x} \Leftrightarrow \frac{1}{2}\ln|y| = \ln|x| + C \Leftrightarrow \ln|y|^{1/2} = \ln|x| + C$$

$$\Leftrightarrow |y|^{1/2} = e^{c} \cdot e^{\ln|x|}$$

$$\Leftrightarrow \sqrt{|y|} = e^{c}|x| \Leftrightarrow |y| = (e^{c})^2 x^2$$

$$\Leftrightarrow \boxed{y = Kx^2}$$

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13.(6 pts.)The solution of the initial value problem

$$xy' = y + x^2 \sin x, \quad y(\pi) = 0$$

is given by

(a) 0

(b) $y = x \sin x$

~~(c)~~ $y = -x(\cos x + 1)$

(d) cannot be determined from the given information

(e) $y = \pi + x \cos x$

$xy' = y + x^2 \sin x$

$\Leftrightarrow xy' - y = x^2 \sin x$

$\Leftrightarrow y' - \frac{1}{x} y = x \sin x$ $\quad P(x) = -\frac{1}{x}$, $G(x) = x \sin x$

Then, $I(x) = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|} = e^{\ln \frac{1}{|x|}} = \frac{1}{|x|} \overset{x>0}{=} \frac{1}{x}$

We then have, $\left(\frac{y}{x}\right)' = \frac{1}{x}(x \sin x)$

$\Leftrightarrow \left(\frac{y}{x}\right)' = \int \sin x \Leftrightarrow \frac{y}{x} = -\cos x + C \Leftrightarrow y = -x\cos x + Cx$

$y(\pi) = 0 \Leftrightarrow 0 = -\pi(-1) + \pi C \Leftrightarrow -\pi C = \pi \Leftrightarrow C = -1.$

So, $y = -x\cos x - x \Leftrightarrow \boxed{y = -x(\cos x + 1)}$

14.(6 pts.)A certain interest rate in the economy, denoted by r, changes with time according to the differential equation

$$\frac{dr}{dt} = 0.1(5 - r).$$

If this rate is equal to 3 today, use Euler's method with a stepsize $h = 2$ to estimate its value in 4 years from now.

(a) 1.5 ~~(b)~~ 3.72 (c) 1.8 (d) 3.5 (e) 3.4

$r(0) = 3$

$r(2) = 3 + 2 \cdot (0.1(5-3)) = 3 + 2 \cdot (0.2) = 3 + 0.4 = 3.4$

$r(4) = 3.4 + 2(0.1(5 - 3.4)) = 3.4 + 2(0.1)(1.6) = 3.4 + (0.2)(1.6)$

$= 3.4 + \frac{2}{10} \cdot \frac{16}{10}$

$= 3.4 + \frac{32}{100}$

$= 3.4 + .32 = \boxed{3.72}$

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15.(6 pts.) Consider the following **sequences**:

$$(I)\ \left\{(-1)^n \frac{n^2-1}{2n^2+1}\right\}_{n=1}^{\infty} \qquad (II)\ \left\{\frac{n^2-1}{e^n}\right\}_{n=1}^{\infty} \qquad (III)\ \left\{2^{1/n}\right\}_{n=1}^{\infty}$$

Which of the following statements is true?

(a) Sequence II diverges and sequences I and III converge.

(b) All three sequences converge.

~~(c)~~ Sequence I diverges and sequences II and III converge.

(d) All three sequences diverge.

(e) Sequence III diverges and sequences I and II converge.

(I) Since $\lim_{n\to\infty} \frac{n^2-1}{2n^2+1} = \frac{1}{2}$, $\lim_{n\to\infty} (-1)^n \frac{n^2-1}{2n^2+1}$ does not exist, so (I) diverges

b/c the terms oscillate between $+\frac{1}{2}$ & $\frac{-1}{2}$ as $n\to\infty$

(II) $\lim_{n\to\infty} \frac{n^2-1}{e^n} = 0$ since e^n goes to ∞ faster than n^2-1; so (II) converges

(III) $\lim_{n\to\infty} 2^{1/n} = L \Leftrightarrow \lim_{n\to\infty} \frac{\ln(2)}{n} = \ln L \Rightarrow 0 = \ln L \Rightarrow L = 1 = \lim_{n\to\infty} 2^{1/n}$.

So, (III) converges.

16.(6 pts.) Find $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{9^{n-1}}$.

(a) $\frac{9}{11}$ (b) $-\frac{18}{7}$ (c) $\frac{4}{3}$ (d) $-\frac{9}{11}$ ~~(e)~~ $\frac{18}{11}$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{9^{n-1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{n-1}\cdot 2}{9^{n-1}} = \sum_{n=1}^{\infty} 2\left(\frac{-2}{9}\right)^{n-1}$$

: geom. series with $r = \frac{-2}{9}$, $a = 2$

$$= 2\cdot\frac{1}{1-\left(\frac{-2}{9}\right)} = 2\cdot\frac{9}{11} = \boxed{\frac{18}{11}}$$

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17.(6 pts.) Consider the following series

$$(I)\ \sum_{n=2}^{\infty} \frac{3n^2+2n+1}{2n^2+n} \qquad (II)\ \sum_{n=2}^{\infty} \frac{n^2}{n^3+1} \qquad (III)\ \sum_{n=1}^{\infty} \frac{3^n}{2(n!)}$$

Which of the following statements is true?

(a) Only I and III converge

(b) Only III converges

(c) All three converge

(d) All three diverge

(e) Only II and III converge

(I) diverges by Test of Divergence : $\lim_{n\to\infty} \frac{3n^2+2n+1}{2n^2+n} = \frac{3}{2} \neq 0$

(II) diverges by Limit Comp Test with $b_n = \frac{n^2}{n^3} = \frac{1}{n}$, $\lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot \frac{n}{1} = 1$

(III) Ratio Test : $\lim_{n\to\infty} \frac{3^{n+1}}{2(n+1)!} \cdot \frac{2(n!)}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1 \Rightarrow$ (III) converges.

18.(6 pts.) Consider the following series

$$(I)\ \sum_{n=3}^{\infty} \frac{\sin(n^2)}{n^2+1} \qquad (II)\ \sum_{n=3}^{\infty} \frac{(-1)^n}{\sqrt{n}-1}.$$

Which of the following statements is true?

(a) (I) is absolutely convergent and (II) is conditionally convergent.

(b) (I) converges and (II) diverges.

(c) (I) and (II) are both conditionally convergent.

(d) (I) and (II) are both absolutely convergent.

(e) (I) and (II) both diverge.

(I) $0 \leq |\sin(n^2)| \leq 1 \Rightarrow \frac{|\sin(n^2)|}{n^2+1} \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$. And since $\sum \frac{1}{n^2}$ converges, $\sum \frac{|\sin(n^2)|}{n^2+1}$ converges by Comp. Test. So, $\sum \frac{\sin(n^2)}{n^2+1}$ converges absolutely.

(II) $\lim_{n\to\infty} \frac{1}{\sqrt{n-1}} = 0$ and since $\sqrt{n} \geq \sqrt{n-1}$, $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n-1}}$. So, $\sum \frac{(-1)^n}{\sqrt{n-1}}$ converges by Alt S. Test. But $\sum \left|\frac{(-1)^n}{\sqrt{n-1}}\right| = \sum \frac{1}{\sqrt{n-1}}$ diverges by Comp. Test to $\sum \frac{1}{\sqrt{n}}$. So, $\sum \frac{(-1)^n}{\sqrt{n-1}}$ converges conditionally.

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19.(6 pts.) Find the radius of convergence R for the power series

$$\sum_{n=1}^{\infty} \frac{(3n+2)(x+1)^n}{5^n(n+1)}.$$

(a) $R = 3/5$ (b) $R = 5/3$ ~~(c)~~ $R = 5$ (d) $R = 3$ (e) $R = 1$

$$\lim_{n\to\infty} \frac{[3(n+1)+2]|x+1|^{n+1}}{5^{n+1}(n+2)} \cdot \frac{5^n(n+1)}{(3n+2)|x+1|^n} = \lim_{n\to\infty} \frac{(3n+5)(n+1)|x+1|}{5(n+2)(3n+2)}$$

$$= |x+1| \lim_{n\to\infty} \frac{3n^2 + \text{"lower terms of } n\text{"}}{15n^2 + \text{"lower terms of } n\text{"}} = |x+1|\left(\frac{3}{15}\right) = \frac{|x+1|}{5} < 1$$

$$\Rightarrow |x+1| < \boxed{5 = R}$$

20.(6 pts.) Find a power series representation for the function

$$\frac{2}{(9-x^2)}$$

in the interval $(-3, 3)$.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2(x^{2n})}{9}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x^{2n})}{9^n}$ (c) $\displaystyle\sum_{n=0}^{\infty} x^{2n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{9(3^n)}$ ~~(e)~~ $\displaystyle\sum_{n=0}^{\infty} \frac{2(x^{2n})}{9^{n+1}}$

$$\frac{2}{9-x^2} = \frac{2}{9}\left[\frac{1}{1-\frac{x^2}{9}}\right] = \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{x^2}{9}\right)^n = \frac{2}{9}\sum_{n=0}^{\infty}\frac{x^{2n}}{9^n} = \boxed{\sum_{n=0}^{\infty}\frac{2x^{2n}}{9^{n+1}}}$$

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21.(6 pts.) Which of the power series series given below is the McLaurin series (i.e. Taylor series at $a = 0$) of

$$\cos(x^2)?$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n+1)!}$. (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(4n)!}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(n)!}$. ~~(e)~~ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$.

We know $\cos x = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, So, $\cos(x^2) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}}$$

22.(6 pts.) Which of the polynomials shown below is the third degree Taylor polynomial of $f(x) = \dfrac{1}{(1-x)^2}$ at $a = -1$?

(a) $1 + 2x + 3x^2 + 4x^3$

(b) $\dfrac{1}{2^2} + \dfrac{2}{2^3}x + \dfrac{3}{2^4}x^2 + \dfrac{4}{2^5}x^3$

(c) $\dfrac{1}{2^2} + \dfrac{2!}{2^3}(x+1) + \dfrac{3!}{2^4}(x+1)^2 + \dfrac{4!}{2^5}(x+1)^3$

~~(d)~~ $\dfrac{1}{2^2} + \dfrac{2}{2^3}(x+1) + \dfrac{3}{2^4}(x+1)^2 + \dfrac{4}{2^5}(x+1)^3$

(e) $1 + (x+1) + (x+1)^2 + (x+1)^3$

	$f^{(n)}(x)$	$f^{(n)}(-1)$
$n=0$	$(1-x)^{-2}$	$\frac{1}{2^2}$
$n=1$	$+2(1-x)^{-3}$	$2 \cdot \frac{1}{2^3}$
$n=2$	$6(1-x)^{-4}$	$6 \cdot \frac{1}{2^4}$
$n=3$	$24(1-x)^5$	$24 \frac{1}{2^5}$

$$T_3(x) = f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3$$

$$= \frac{1}{2^2} + \frac{2}{2^3}(x+1) + \frac{6}{2^4} \cdot \frac{1}{2!}(x+1)^2 + \frac{24}{2^5}\frac{1}{\underbrace{3!}_{=6}}(x+1)^3$$

$$= \boxed{\frac{1}{2^2} + \frac{2}{2^3}(x+1) + \frac{3}{2^4}(x+1)^2 + \frac{4}{2^5}(x+1)^3}$$

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23.(6 pts.) Which line below is the tangent line to the parameterized curve

$$x = \cos t + 2\cos(2t), \qquad y = \sin t + 2\sin(2t)$$

when $t = \pi/2$?

(a) $y = -x + 3$

~~(b)~~ $y = 4x + 9$

(c) $y = -4x - 7$

(d) $y = x + 3$

(e) $y = 1$

$$\frac{dy}{dx} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/2} = \left.\frac{\cos t + 4\cos(2t)}{-\sin t - 4\sin(2t)}\right|_{t=\pi/2} = \frac{0 + 4(-1)}{-1 - 4\cdot 0} = \frac{-4}{-1} = 4$$

$x(\frac{\pi}{2}) = 0 + 2(-1) = -2$; $y(\frac{\pi}{2}) = 1 + 2\cdot 0 = 1$

Eq of tangent line : $y - 1 = 4(x-(-2)) = y = 4x + 8 + 1 \Leftrightarrow \boxed{y = 4x+9}$

24.(6 pts.) Which integral below computes the length of the parameterized curve

$$x(t) = 1 + e^{2t}, \qquad y(t) = \sin(2t)$$

for $0 \le t \le 1$?

(a) $\int_0^1 \sqrt{2e^{2t} + 2\cos(2t)}\, dt$

(b) $\int_0^1 \sqrt{(1+e^{2t})^2 + \sin^2(2t)}\, dt$

~~(c)~~ $\int_0^1 \sqrt{4e^{4t} + 4\cos^2(2t)}\, dt$

(d) $\int_0^1 \sqrt{1 + \sin^2(2t)}\, dt$

(e) $\int_0^1 \sqrt{(1+e^{2t}) + \sin(2t)}\, dt$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{\left(2e^{2t}\right)^2 + \left(2\cos(2t)\right)^2}\, dt$$

$$= \boxed{\int_0^1 \sqrt{4e^{4t} + 4\cos^2(2t)}\, dt}$$

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25.(6 pts.) Find the area of the region enclosed by the polar curve $r = \cos(2\theta)$, $0 \leq \theta \leq 2\pi$.
(Note: The formula sheet may help here.)

(a) 2π (b) 2 ~~(c)~~ $\frac{\pi}{2}$ (d) $\frac{\pi^2}{2}$ (e) π

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cos^2(2\theta) \, d\theta = \int_0^{2\pi} \frac{1}{2} \left(\frac{1+\cos 4\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} (1 + \cos 4\theta) \, d\theta = \frac{1}{4} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{2\pi} = \frac{1}{4} \left[2\pi \right] = \boxed{\frac{\pi}{2}}$$

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The following is the list of useful trigonometric formulas:

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x-y) + \sin(x+y)\big)$$

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\int \csc\theta d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$\int \csc^2\theta d\theta = -\cot x + C$$